Gauss - Codazzi equations for generic surfaces: equivalence to the DS linear problem with constraint, linearizability and reductions

1997 J. Phys. A: Math. Gen. 30 L437

(http://iopscience.iop.org/0305-4470/30/13/003)

J. Phys. A: Math. Gen. **30** (1997) L437–L441. Printed in the UK PII: S0305-4470(97)82405-5

LETTER TO THE EDITOR

Gauss–Codazzi equations for generic surfaces: equivalence to the DS linear problem with constraint, linearizability and reductions

B G Konopelchenko

Dipartimento di Fisica, Universita di Lecce, 73100 Lecce, Italy
and
Budker Institute of Nuclear Physics, Novosibirsk, Russia

Received 7 March 1997

Abstract. It is shown that the standard Gauss–Codazzi system of equations which describe generic surfaces immersed into the three-dimensional Euclidean space is equivalent to the two-dimensional Dirac equation (Davey–Stewartson linear problem) accompanied by a specific constraint. It is demonstrated that this system is linearizable via the parametrization of the moving trihedral by the Euler angles. Some special classes of surfaces are also considered.

The deep interrelation between many nonlinear differential equations of the classical differential geometry of surfaces (see e.g. [1–3]) and modern soliton equations (see e.g. [4, 5]) is well established now. Since the famous sine–Gordon equation $\phi_{xy} = \sin\phi$ and the Liouville equation $\phi_{xy} = \exp\phi$ this interrelation has been studied from various points of view in numerous papers. Most of these papers were devoted to the study of special classes of surfaces or surfaces referred to special coordinates and consequently to special classes of associated nonlinear partial differential equations.

In the present paper, in contrast we will consider generic surfaces immersed into the three-dimensional Euclidean space $\mathbb{R}^3$. We will demonstrate that the Gauss–Codazzi (GC) equations for generic surfaces referred to their curvature lines are equivalent to a simple system which consists of the two-dimensional Dirac equation (Davey–Stewartson (DS)-I linear problem) and a specific constraint. It is shown that this constraint is associated with the stationary point of the special symmetry of the DS-I linear problem. We demonstrate that the generic GC system is linearizable. The parametrization of the moving trihedral by the Euler angles is an appropriate tool. We also discuss solutions, reductions and other properties of the GC system.

1. We consider a generic surface immersed into the three-dimensional Euclidean space $\mathbb{R}^3$. It is well known that everywhere outside umbilic points the first and second fundamental forms Ω_1 and Ω_2 can be diagonalized simultaneously (see e.g. [3]). Thus choosing the curvature lines as the coordinate lines, one, generically, has

$$\begin{aligned}\Omega_1 &= g_{11}\,\mathrm{d}x^2 + g_{22}\,\mathrm{d}y^2\\ \Omega_2 &= d_{11}\,\mathrm{d}x^2 + d_{22}\,\mathrm{d}y^2.\end{aligned} \tag{1}$$

In these coordinates the standard GC equations take the form (see [3])

$$\left(\frac{d_{11}}{\sqrt{g_{11}}}\right)_y - \frac{d_{22}}{g_{22}}(\sqrt{g_{11}})_y = 0 \qquad \left(\frac{d_{22}}{\sqrt{g_{22}}}\right)_x - \frac{d_{11}}{g_{11}}(\sqrt{g_{22}})_x = 0$$
$$\left(\frac{1}{\sqrt{g_{11}}}(\sqrt{g_{22}})_x\right)_x + \left(\frac{1}{\sqrt{g_{22}}}(\sqrt{g_{11}})_y\right)_y + \frac{d_{11}d_{22}}{\sqrt{g_{11}g_{22}}} = 0. \tag{2}$$

We now denote

$$\psi_1 = \frac{d_{11}}{\sqrt{g_{11}}} \qquad \psi_2 = \frac{d_{22}}{\sqrt{g_{22}}} \tag{3}$$

and

$$p = \frac{1}{\sqrt{g_{22}}}(\sqrt{g_{11}})_y \qquad q = \frac{1}{\sqrt{g_{11}}}(\sqrt{g_{22}})_x. \tag{4}$$

In such notations equations (2) look like

$$\psi_{1y} = p\psi_2 \qquad \psi_{2x} = q\psi_1 \tag{5a}$$
$$q_x + p_y + \psi_1\psi_2 = 0. \tag{5b}$$

Denoting $\sqrt{g_{11}} = \widetilde{\psi}_1$, $\sqrt{g_{22}} = \widetilde{\psi}_2$ and using the definition (4), one also has

$$\widetilde{\psi}_{1y} = p\widetilde{\psi}_2 \qquad \widetilde{\psi}_{2x} = q\widetilde{\psi}_1. \tag{6}$$

In the form (5) the GC equations represent the two-dimensional Dirac equation or, in soliton terminology, the DS-I linear problem (5*a*) with the constraint (5*b*). Note that the DS-I linear problem (5*a*) represents the Codazzi equations while the constraint (5*b*) is the Gauss equation. We will see that such a form of the GC equation is useful from different points of view. The systems of equations of the type (5) are not completely new. Similar systems have been discussed in [6] within the study of the so-called orthogonal nets.

The systems (5) and (6) provide us with the following way of constructing the surfaces. First one has to solve the system (5). Then taking p and q as found, one finds $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ by solving (6). As a result one gets the three fundamental forms of the surface

$$\Omega_1 = \widetilde{\psi}_1^2\,\mathrm{d}x^2 + \widetilde{\psi}_2^2\,\mathrm{d}y^2 \qquad \Omega_2 = \widetilde{\psi}_1\psi_1\,\mathrm{d}x^2 + \widetilde{\psi}_2\psi_2\,\mathrm{d}y^2 \qquad \Omega_3 = \psi_1^2\,\mathrm{d}x^2 + \psi_2^2\,\mathrm{d}y^2. \tag{7}$$

and, according to the well known theorem (see e.g. [1–3]), a surface is determined to within a motion in space. In particular, the principal curvatures are

$$K_1 = \frac{\psi_1}{\widetilde{\psi}_1} \qquad K_2 = \frac{\psi_2}{\widetilde{\psi}_2} \tag{8}$$

and the total curvature K is equal to

$$K = \iint_S K_1K_2\sqrt{\det g}\,\mathrm{d}x\wedge\mathrm{d}y = \iint_S \psi_1\psi_2\,\mathrm{d}x\wedge\mathrm{d}y = -\iint_S (q_x + p_y)\,\mathrm{d}x\wedge\mathrm{d}y. \tag{9}$$

Note that the umbilic points ($K_1 = K_2$) correspond to linearly dependent solutions

$$\psi_i, \widetilde{\psi}_i \left(\begin{vmatrix} \psi_1 & \widetilde{\psi}_1 \\ \psi_2 & \widetilde{\psi}_2 \end{vmatrix} = 0\right).$$

The constraint (5*b*) (Gauss equation) is crucial for the geometric interpretation of the system (5). It picks up those solutions of the DS-I linear problem (5*a*) and (6) for which $\widetilde{\psi}_i^2$, $\widetilde{\psi}_i\psi_i$, ψ_i^2 ($i = 1, 2$) are components of the fundamental forms (7) of some surface. The constant (5*b*) is connected with the special symmetry of the linear system (5*a*). Indeed, one can show that the DS-1 linear problem (5*a*) is form-invariant under transformations $(p, q, \psi_1, \psi_2) \to (p', q', \psi_1', \psi_2')$ given by the relations

$$\psi_1' = \psi_{1,x}^* - p'\psi_2^* + \psi_1'\phi \qquad \psi_2' = -\psi_{2,y}^* + q'\psi_1^* + \psi_2'\phi \qquad q_x + p_y' + \psi_1'\psi_2 = 0$$
$$q_x' + p_y + \psi_1\psi_2' = 0 \qquad q'p' = qp \tag{10}$$

where ψ_1^*, ψ_2^* are the solutions of the formally adjoint DS-I linear problem

$$\psi_{1y}^* = -q\psi_2^* \qquad \psi_{2x}^* = -p\psi_1^* \tag{11}$$

and ϕ is defined by $\phi_x = \psi_1\psi_1^*$, $\phi_y = -\psi_2\psi_2^*$. The stationary point of the transformation (10) is characterized by the conditions

$$\psi_1 = \psi_{1,x}^* - p\psi_2^* + \psi_1\phi \qquad \psi_2 = -\psi_{2,y}^* + q\psi_1^* + \psi_2\phi \qquad q_x + p_y + \psi_1\psi_2 = 0. \tag{12}$$

Thus, the constraint (5*b*) selects those solutions of the DS-I linear problem (5*a*) which are invariant under transformation (10). More generally, the constraint (5*b*) picks up the solutions of the system (5*a*) for which the combinations

$$\chi_1 = \psi_{1,x}^* - p\psi_2^* + \psi_1\phi \qquad \chi_2 = -\psi_{2,y}^* + q\psi_1^* + \psi_2\phi$$

solve the same system (5*a*) as ψ_1 and ψ_2.

2. Let us now consider some special classes of surfaces. The simplest example is given by $\psi_1 = \alpha\sin\theta$, $\psi_2 = \alpha\cos\theta$, $p = \theta_y, q = -\theta_x$ for which the system (5*a*) is satisfied identically while equation (5*b*) becomes the well known sine–Gordon equation $\theta_{xx} - \theta_{yy} = \frac{1}{2}\alpha^2\sin\theta$. Choosing $\tilde\psi_1 = \tilde\alpha\sin\theta$, $\tilde\psi = \tilde\alpha\cos\theta$, we get $K_1 = K_2 = \alpha/\tilde\alpha$. It is the well known case of the round sphere.

For the isotermic surfaces $g_{11} = g_{22}$ (see [1–3]). So $p = \theta_y, q = \theta_x$ where $\theta = \frac{1}{2}\log g_{11}$ and the system (5) is of the form

$$\psi_1 = \theta_y\psi_2 \qquad \psi_2 = \theta_x\psi_1 \tag{13a}$$
$$\theta_{xx} + \theta_{yy} + \psi_1\psi_2 = 0. \tag{13b}$$

In the terms of the principal curvatures $K_1 = \psi_1\exp(-\theta)$, $K_2 = \psi_2\exp(-\theta)$ the system (13) has the well known form [1–3]. It has recently been studied in [7]. Choosing $\psi_1 = \alpha\exp\theta + \beta\exp(-\theta), \psi_2 = \alpha\exp\theta - \beta\exp(-\theta)$ where α, β are arbitrary constants, one satisfies equations (13*a*) identically while equation (13*b*) takes the form

$$\theta_{xx} + \theta_{yy} + \alpha^2\exp 2\theta - \beta^2\exp(-2\theta) = 0 \tag{14}$$

which is the well known equation (the Liouville equation at $\beta = 0$ and sinh–Gordon equation for $\alpha \neq 0$, $\beta \neq 0$). Taking $\tilde\psi_1 = \tilde\alpha\exp\theta + \tilde\beta\exp(-\theta)$ and $\tilde\psi_2 = \tilde\alpha\exp\theta - \tilde\beta\exp(-\theta)$ where $\tilde\alpha, \tilde\beta$ are arbitrary constants and θ is a solution of equation (14), one gets a family of isotermic surfaces with the fundamental forms and principal curvatures given by (7) and (8). The Gaussian curvature K and mean curvature H are of the form

$$K = \frac{\alpha^2\exp 2\theta - \beta^2\exp(-2\theta)}{\tilde\alpha^2\exp 2\theta - \tilde\beta^2\exp(-2\theta)} \qquad H = \frac{\alpha\tilde\alpha\exp 2\theta - \beta\tilde\beta\exp(-2\theta)}{\tilde\alpha^2\exp 2\theta - \tilde\beta^2\exp(-2\theta)}.$$

Choosing the constants α, β, $\tilde\alpha$, $\tilde\beta$, one gets various particular cases. For instance, at $\alpha = \beta$ and $\tilde\beta = 0$ one has the constant mean curvature surfaces with $H = \alpha/\tilde\alpha$ and $K = (\alpha^2/\tilde\alpha^2)(1 - \exp(-4\theta))$.

The third special case is given by the constraint $q = 1$. The system (5) takes the form

$$\varphi_{xy} - p\varphi = 0 \qquad p_y + \tfrac{1}{2}(\varphi^2)_x = 0$$

where $\varphi = \psi_2$. The integrability of this system has been demonstrated in [8]. In fact, it is equivalent to the sine–Gordon equation [9].

3. The GC system is the compability condition for the linear Gauss–Weingarten–Mainardi–Peterson (GWMP) equations (see [1–3]). In the terms of p, q, ψ_1 and ψ_2, the GWMP equations for generic surfaces look like

$$\chi_x = (\psi_1 S_2 + p S_3)\chi \qquad \chi_y = (-\psi_2 S_1 - q S_3)\chi \tag{15}$$

where χ is the 3×3 orthogonal matrix the first two lines of which are formed from the cosine-directions of the tangent vectors and the third line is formed by the three components of the normal vector. The matrices S_1, S_2, S_3 are of the form $(S_i)_{kl} = -\varepsilon_{ikl}$ $(i, k, l = 1, 2, 3)$ and ε_{ikl} is a totally antisymmetric tensor $(\varepsilon_{123} = 1)$. Matrices S_i obey the commutation relations $[S_i, S_k] = \varepsilon_{ikl} S_l$ $i, k = 1, 2, 3$ and they form a basic of the algebra so(3). Using the isomorphism between so(3) and su(2), one gets the simplest spinor form of the system (15):

$$\phi_x = \tfrac{1}{2}\mathrm{i}(\psi_1 \sigma_2 + p\sigma_3)\phi \qquad \phi_y = -\tfrac{1}{2}\mathrm{i}(\psi_2 \sigma_1 + q\sigma_3)\phi \tag{16}$$

where σ_1, σ_2, σ_3 are the standard Pauli matrices $(S_k = (\tfrac{1}{2}\mathrm{i})\sigma_k)$.

The linear system (16) contains no spectral parameter. Nevertheless, as we will see, it is a rather useful tool with which to analyse the system (5). First we note for real-valued p, q, ψ_1 and ψ_2 equation (16) implies that $\phi^+\phi = A$ where A is a constant positively defined matrix. So without loss of generality one can consider solutions ϕ of the system (16) as the unitary 2×2 matrices $(\phi^+\phi = 1)$.

From (16) one has

$$\psi_1\sigma_2 + p\sigma_3 = 2\mathrm{i}\phi_x\phi^+ \qquad \psi_2\sigma_1 + q\sigma_3 = -2\mathrm{i}\phi_y\phi^+. \tag{17}$$

Equations (16) imply that any unitary 2×2 matrix ϕ such that the $\phi_x\phi^+$ and $\phi_y\phi^+$ have the form of the left-hand side of (17) provides via (17) the solution of the GC system (5). To describe such matrices we note that as the result of the property $\mathrm{tr}(\phi_x\phi^+) = \mathrm{tr}(\phi_y\phi^+) = 0$, this requirement is equivalent to the following conditions

$$\mathrm{tr}(\sigma_1\phi_x\phi^+) = 0 \qquad \mathrm{tr}(\sigma_2\phi_y\phi^+) = 0. \tag{18}$$

So any 2×2 unitary matrix ϕ which obeys the constraints (18) provides via (17) a solution of the GC system (5).

To solve the constraints (18) we use the well-known parametrization of the unitary matrix

$$\phi = \begin{pmatrix} \cos\beta/2\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha+\gamma)} & -\sin\beta/2\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\alpha-\gamma)} \\ \sin\beta/2\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha-\gamma)} & \cos\beta/2\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\alpha+\gamma)} \end{pmatrix} \tag{19}$$

by the Euler angles α, β, γ (see e.g. [10]). Here α, β, γ are real-valued functions of x and y. Straightforward calculation shows that in such a parametrization the constraints (18) take the form

$$(\log \mathrm{tg}\,\beta/2)_x = \mathrm{ctg}\,\alpha \cdot \gamma_x \qquad (\log \mathrm{tg}\,\beta/2)_y = -\mathrm{tg}\,\alpha \cdot \gamma_y. \tag{20}$$

On the other hand, the use of the formula (17) gives

$$p = \alpha_x + \cos\beta \cdot \gamma_x \qquad q = -\alpha_y - \cos\beta \cdot \gamma_y \qquad \psi_1 = -\frac{\sin\beta}{\sin\alpha} \cdot \gamma_x$$
$$\psi_2 = \frac{\sin\beta}{\cos\alpha} \cdot \gamma_y. \tag{21}$$

So, any three functions α, β, γ which obey the system of equations (20) provide via (21) a solution of the GC system (5). It is easy to check this fact straightforwardly. Note that equation (20) implies

$$\gamma_{xy} = (\log\sin\alpha)_y \cdot \gamma_x + (\log\cos\alpha)_x \cdot \gamma_y \tag{22}$$

and a similar equation for $V = \log \operatorname{tg} \beta/2$. It is important that the system (20) contains the function α algebraically and there is no constraint on α. Thus the formulae (20) and (22) provide us with the linearization of the GC system (5): first, one chooses an arbitrary function $\alpha(x, y)$, second, one solves the linear Laplace equation (22), then, integrating the right-hand side of (20), one finds β and, finally, the formulae (21) give us a solution of equations (5). Note that the parametrization of the moving trihedral by the Euler angles has been used by Bonnet [11] for the triply orthogonal systems of surfaces.

So, the generic GC system is the C-integrable nonlinear system (linearizable by change of variables) in contrast to some its special cases (like the sine–Gordon equations) which are S-integrable systems (integrable by the inverse spectral transform method).

4. The linearization (21) and (22) provides us with the way to construct solutions of the system (5). Indeed, let us start with $\alpha = \alpha_0 =$ constant. The general solution of equation (22) in this case is $\gamma = a(x) + b(y)$ where a and b are arbitrary functions. Then (20) gives $\tan \frac{1}{2}\beta = \exp[-\cot\alpha_0 \cdot a(x) + \tan\alpha_0 \cdot b(y)]$. Finally, one gets

$$p = -\tan\alpha_0 (\log\cos z)_x \qquad q = -\cot\alpha_0 (\log\cos z)_y$$
$$\psi_1 = \frac{2}{\cos\alpha_0}[\log(1+\exp z)]_x \qquad \psi_2 = \frac{2}{\sin\alpha_0}[\log(1+\exp z)]_y \tag{23}$$

where $z = -\cot\alpha_0 \cdot a(x) + \tan\alpha_0 \cdot b(y)$. To proceed we can use the Darboux type transformation for equation (22) found within a different context in [12]:

$$\gamma \to \gamma' = \gamma - \tilde{\gamma}\frac{M(\tilde{\gamma}, \gamma)}{M(\tilde{\gamma}, \tilde{\gamma})} \qquad \alpha \to \alpha' = \alpha + 2\arctan\frac{\gamma_x}{\gamma_y}$$

where $\tilde{\gamma}$ is an another solution of (22) and $M(\varphi, \chi) = \varphi_x\chi_y - \varphi_y\chi_x$.

The linearizability of the GC equations via the transformation (21) breaks down for special (non-generic) classes of surfaces. For instance, for the sine–Gordon reduction ($p_x + q_y = 0$) one has an additional equation

$$\alpha_{xx} - \alpha_{yy} + (\cos\beta \cdot \gamma_x)_x - (\cos\beta \cdot \gamma_y)_y = 0.$$

For the isometric surfaces ($p_x - q_y = 0$) the constraint is

$$\alpha_{xx} + \alpha_{yy} + (\cos\beta \cdot \gamma_x)_x + (\cos\beta \cdot \gamma_y)_y = 0.$$

The forms (5) and (20) of the GC equations are useful for an analysis of other properties of surfaces too.

The author is grateful to B Dubrovin for useful discussion.

References

[1] Darboux G 1877–1896 *Lecons sur la théorie des surfaces et les applications geometriques du calcul infinitesimal* vol 1–4 (Paris: Gauthier-Villars)
[2] Bianchi L 1902 *Lezioni di geometria differentiable* 2nd edn (Piza: Spoerri)
[3] Eisenhart L P 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (New York: Dover)
[4] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1980 *Theory of Solitons* (Moscow: Nauka) 1984 (Consultant Bureau)
[5] Ablowitz M and Segur H 1980 *Solitons and Inverse Scattering Transform* (Philidelphia: SIAM)
[6] Eisenhart L P 1962 *Tranformations of Surfaces* (New York: Chelsea)
[7] Cieslinski J, Goldstein P and Sym A 1995 *Phys. Lett.* **205A** 37
[8] Konopelchenko B G 1992 *Introduction to Multidimensional Integrable Equations* (New York: Plenum)
[9] Hirota R and Tsujimoto S 1994 *J. Phys. Soc. Japan* **63** 3533
[10] Wigner E P 1931 *Gruppentheorie* (Braunschweig: Friedr. Vieweg and Son)
[11] Bonnet O 1862 *Comp. Rend.* **LIV** 554
[12] Konopelchenko B G, Schief W and Rogers C 1992 *Phys. Lett.* **172A** 39